

# THE AVERAGE OF THE SMALLEST PRIME IN A CONJUGACY CLASS

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ABSTRACT. Let  $C$  be a conjugacy class of  $S_n$  and  $K$  an  $S_n$ -field. Let  $n_{K,C}$  be the smallest prime which is ramified or whose Frobenius automorphism  $\text{Frob}_p$  does not belong to  $C$ . Under some technical conjectures, we compute the average of  $n_{K,C}$ . For  $S_3$  and  $S_4$ -fields, our result is unconditional. For  $S_n$ -fields,  $n = 3, 4, 5$ , we give a different proof which depends on the strong Artin conjecture. Let  $N_{K,C}$  be the smallest prime for which  $\text{Frob}_p$  belongs to  $C$ . For  $S_3$ -fields, we obtain an unconditional result for the average of  $N_{K,C}$  for  $C = [(12)]$ .

## 1. INTRODUCTION

For a fundamental discriminant  $D$ , let  $\chi_D(\cdot) = \left(\frac{D}{\cdot}\right)$ , and let  $N_{D,\pm 1}$  be the smallest prime such that  $\chi_D(p) = \pm 1$ , resp. Let  $n_{D,\pm 1}$  be the smallest prime such that  $\chi_D(p) \neq \pm 1$ . We can interpret  $N_{D,1}$  ( $N_{D,-1}$ ) as the smallest prime which splits completely (inert, resp.) in a quadratic field  $\mathbb{Q}(\sqrt{D})$ . Under the assumption of the Generalized Riemann Hypothesis (GRH) for  $L(s, \chi_D)$ , one can show easily that  $N_{D,\pm 1}, n_{D,\pm 1} \ll (\log D)^2$ . Erdős [13] considered the average of those values over a family:  $\lim_{X \rightarrow \infty} \frac{\sum_{2 < p \leq X} N_{p,-1}}{\pi(X)} = \sum_{k=1}^{\infty} \frac{p_k}{2^k} = 3.67464\dots$  where  $p$  runs through primes, and  $p_k$  is the  $k$ -th prime. Pollack<sup>1</sup> [24] generalized Erdős' result to all fundamental discriminants:

$$\lim_{X \rightarrow \infty} \frac{\sum_{|D| \leq X} N_{D,\pm 1}}{\sum_{|D| \leq X} 1} = \sum_q \frac{q^2}{2(q+1)} \prod_{p < q} \frac{p+2}{2(p+1)} = 4.98094\dots$$

Pollack [23] also computed the average of the least inert primes over cyclic number fields of prime degree.

We generalize this problem to the setting of general number fields. We call a number field  $K$  of degree  $n$ , an  $S_n$ -field if its Galois closure  $\widehat{K}$  over  $\mathbb{Q}$  is an  $S_n$  Galois extension. Let  $C$  be a conjugacy class of  $S_n$ . For an unramified prime  $p$ , denote by  $\text{Frob}_p$ , a Frobenius automorphism

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<sup>1</sup>In [20], the value 4.98094.. is misquoted as 4.98085.

of  $p$ . Define  $n_{K,C}$  to be the smallest prime  $p$  which is ramified in  $K$  or for which  $\text{Frob}_p \notin C$ , and define  $N_{K,C}$  to be the smallest prime  $p$  such that  $\text{Frob}_p \in C$ . Under GRH, we can show that  $n_{K,C}, N_{K,C} \leq (\log |d_K|)^2$  (cf. [3]).

In this paper we consider the average value of  $n_{K,C}$  over fields in  $L_n^{(r_2)}(X)$ , which is the set of  $S_n$ -fields  $K$  of signature  $(r_1, r_2)$  with  $|d_K| \leq X$ , where  $d_K$  is the discriminant of  $K$ :

**Theorem 1.1.** *Let  $n = 3, 4, 5$ . When  $n = 5$ , we assume either the strong Artin conjecture, or Conjecture 4.1. Then,*

$$(1.2) \quad \frac{1}{|L_n^{(r_2)}(X)|} \sum_{K \in L_n^{(r_2)}(X)} n_{K,C} = \sum_q \frac{q(1 - |C|/|S_n| + f(q))}{1 + f(q)} \prod_{p < q} \frac{|C|/|S_n|}{1 + f(p)} + O\left(\frac{1}{\log X}\right).$$

For  $S_3$ -fields, Martin and Pollack [20] computed (1.2) for  $C = e, [(123)]$ . The main key ingredient was counting  $S_3$ -fields with finitely many local conditions, which is a recent result of Taniguchi and Thorne [27]. In [8], we were able to count  $S_4$  and  $S_5$ -fields with finitely many local conditions using a result of Belabas, Bhargava and Pomerance [1], and a result of Shankar and Tsimerman [26].

Our key idea is to use the unique quadratic subextension  $F = \mathbb{Q}[\sqrt{d_K}]$ , which we call the quadratic resolvent. For unconditional bounds on  $n_{K,C}$ , we use the inequality  $n_{K,C} \leq n_{F,1} \leq N_{F,-1}$  or  $n_{K,C} \leq n_{F,-1} \leq N_{F,1}$  depending on whether  $C \subset A_n$  or  $C \not\subset A_n$ .

We have unconditional bounds of  $N_{F,\pm 1}$  by Norton [21] and Pollack [22]. We review this in Section 3.1.

By using the zero-free region of  $L(s, \chi_F)$ , we can get conditional bounds on  $n_{K,C}$ . This is done in Section 3.2. We need to count the number of  $S_n$ -fields with the same quadratic resolvent. For  $S_3$ -fields, we can estimate such numbers by using the result of [9]. This is done in Section 4. But for  $n \geq 4$ , we do not have such a result. So we state it as Conjecture 4.1. In the Appendix, using the result in [10], we count the number of  $S_4$ -fields with the given cubic resolvent.

In section 5, we establish (1.2) under the counting conjectures (2.2) – (2.3) and Conjecture 4.1. See Theorem 5.2 and tables below it for the average values which were computed using PARI.

In Section 2, we explain counting number fields with finitely many local conditions, which is the main tool for the proof.

In Section 6, we give another proof for  $S_n$ -fields,  $n = 3, 4, 5$ , in order to avoid using Conjecture 4.1. We use the strong Artin conjecture and zero-free regions of different Artin  $L$ -functions for each conjugacy class.

In Section 7, we consider the average of  $N_{K,C}$ , the smallest prime  $p$  with  $\text{Frob}_p \in C$ . In Martin and Pollack [20], the average of  $N_{K,C}$  for  $S_3$ -fields was studied under GRH for the Dedekind zeta functions. We generalize their result and compute the averages of  $N_{K,C}$  under GRH for Dedekind zeta functions and the counting conjectures (2.2) – (2.3). Tables for the average values for  $S_3$ ,  $S_4$ , and  $S_5$  are provided.

In Section 8, we obtain an unconditional result for the average of  $N_{K,C'}$  under Conjecture 4.1 where  $C'$  is the union of all the conjugacy classes not in  $A_n$ . Hence in particular, for  $S_3$ -fields and  $C = [(12)]$ , we have an unconditional result for the average of  $N_{K,C}$ .

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## 2. COUNTING NUMBER FIELDS WITH LOCAL CONDITIONS

Let  $K$  be a  $S_n$ -field for  $n \geq 3$ . Let  $\mathcal{S} = (\mathcal{LC}_p)$  be a finite set of local conditions:  $\mathcal{LC}_p = S_{p,C}$  means that  $p$  is unramified and the conjugacy class of  $\text{Frob}_p$  is  $C$ . Define  $|\mathcal{S}_{p,C}| = \frac{|C|}{|S_n|(1+f(p))}$  for some function  $f(p)$  which satisfies  $f(p) = O(\frac{1}{p})$ . There are also several splitting types of ramified primes, which are denoted by  $r_1, r_2, \dots, r_w$ :  $\mathcal{LC}_p = S_{p,r_j}$  means that  $p$  is ramified and its splitting type is  $r_j$ . We assume that there are positive valued functions  $c_1(p), c_2(p), \dots, c_w(p)$  with  $\sum_{i=1}^w c_i(p) = f(p)$  and define  $|\mathcal{S}_{p,r_i}| = \frac{c_i(p)}{1+f(p)}$ . Let  $|\mathcal{S}| = \prod_p |\mathcal{LC}_p|$ .

**Conjecture 2.1.** *Let  $L_n^{(r_2)}(X; \mathcal{S})$  be the set of  $S_n$ -fields  $K$  of signature  $(r_1, r_2)$  with  $|d_K| < X$  and the local conditions  $\mathcal{S}$ .*

$$(2.2) \quad |L_n^{(r_2)}(X)| = A(r_2)X + O(X^\delta),$$

$$(2.3) \quad |L_n^{(r_2)}(X; \mathcal{S})| = |\mathcal{S}|A(r_2)X + O\left(\left(\prod_{p \in \mathcal{S}} p\right)^\gamma X^\delta\right)$$

for some positive constant  $\delta < 1$  and  $\gamma$ , and the implied constant is uniformly bounded for  $p$  and local conditions at  $p$ .

This conjecture is true for  $S_3, S_4$  and  $S_5$ -fields. For  $S_3$ -fields, we use a result of Taniguchi and Thorne [27]. Let  $f(p) = p^{-1} + p^{-2}$ . Put

$$|S_{p,r_j}| = \frac{p^{-1}}{1+f(p)}, \frac{p^{-2}}{1+f(p)},$$

for  $r_j = (1^2 1), (1^3)$ , respectively. Then

**Theorem 2.4.** [27] Let  $D_0 = \frac{C_1 C^0}{12\zeta(3)}$ ,  $D_1 = \frac{C_1 C^1}{12\zeta(3)}$ .

$$|L_3^{(r_2)}(X, \mathcal{S})| = |\mathcal{S}| D_{r_2} X + O_\epsilon \left( \left( \prod_{p \in \mathcal{S}} p \right)^{\frac{16}{9}} X^{\frac{7}{9} + \epsilon} \right).$$

For  $S_4$ -fields, take  $f(p) = p^{-1} + 2p^{-2} + p^{-3}$ . For a conjugacy class  $C$  of  $S_4$ , let

$$|S_{p,C}| = \frac{|C|}{24(1 + f(p))}.$$

Put

$$|S_{p,r_j}| = \frac{1/2 \cdot 1/p}{1 + f(p)}, \frac{1/2 \cdot 1/p}{1 + f(p)}, \frac{1/2 \cdot 1/p^2}{1 + f(p)}, \frac{1/2 \cdot 1/p^2}{1 + f(p)}, \frac{1/p^2}{1 + f(p)}, \text{ and } \frac{1/p^3}{1 + f(p)}$$

for  $r_j = (1^2 11), (1^2 2), (1^2 1^2), (2^2), (1^3 1), (1^4)$ , respectively. By using the results of [1], [28], we showed

**Theorem 2.5.** [8] Let  $D_i = d_i \prod_p (1 + p^{-2} - p^{-3} - p^{-4})$ , and  $d_0 = \frac{1}{48}, d_1 = \frac{1}{8}$ , and  $d_0 = \frac{1}{16}$ .

$$|L_4^{(r_2)}(X, \mathcal{S})| = |\mathcal{S}| D_{r_2} X + O_\epsilon \left( \left( \prod_{p \in \mathcal{S}} p \right)^2 X^{\frac{143}{144} + \epsilon} \right).$$

For  $S_5$ -fields, take  $f(p) = p^{-1} + 2p^{-2} + 2p^{-3} + p^{-4}$ . For a conjugacy class  $C$  of  $S_5$ , let

$$|S_{p,C}| = \frac{|C|}{120(1 + f(p))}.$$

Put

$$|S_{p,r_j}| = \frac{1/6 \cdot 1/p}{1 + f(p)}, \frac{1/2 \cdot 1/p}{1 + f(p)}, \frac{1/3 \cdot 1/p}{1 + f(p)}, \frac{1/2 \cdot 1/p^2}{1 + f(p)}, \frac{1/2 \cdot 1/p^2}{1 + f(p)}, \frac{1/2 \cdot 1/p^2}{1 + f(p)}, \frac{1/2 \cdot 1/p^2}{1 + f(p)},$$

$$\frac{1/p^3}{1 + f(p)}, \frac{1/p^3}{1 + f(p)}, \text{ and } \frac{1/p^4}{1 + f(p)}$$

for  $r_j = (1^2 111), (1^2 12), (1^2 3), (1^2 1^2 1), (2^2 1), (1^3 11), (1^3 2), (1^3 1^2), (1^4 1), (1^5)$ , respectively.

By using the result of [26], we showed

**Theorem 2.6.** [8] Let  $D_i = d_i \prod_p (1 + p^{-2} - p^{-4} - p^{-5})$  and  $d_0, d_1, d_2$  are  $\frac{1}{240}, \frac{1}{24}$  and  $\frac{1}{16}$ , respectively.

$$|L_5^{(r_2)}(X, \mathcal{S})| = |\mathcal{S}| D_{r_2} X + O_\epsilon \left( \left( \prod_{p \in \mathcal{S}} p \right)^{2-\epsilon} X^{\frac{199}{200} + \epsilon} \right).$$

3. BOUNDS ON  $n_{K,C}$ 

Recall that  $n_{K,C}$  is the smallest prime which is ramified in  $K$  or for which  $\text{Frob}_p$  does not belong to  $C$ . Now  $\widehat{K}$  has the quadratic field  $F$  fixed by  $A_n$ , i.e.,  $F = \mathbb{Q}[\sqrt{d_K}]$ . Let  $d_F$  be the discriminant of  $F$ . Then clearly,  $|d_F| \leq |d_K|$ . By abuse of language, we call such  $F$  the quadratic resolvent of  $K$ .

If  $C \subset A_n$  and  $\text{Frob}_p \in C$ , then  $p$  splits in  $F$ . Hence  $n_{K,C} \leq n_{F,1}$ .

If  $C \not\subset A_n$  and  $\text{Frob}_p \in C$ , then  $p$  is inert in  $F$ . Hence  $n_{K,C} \leq n_{F,-1}$ ,

**3.1. Unconditional bounds of  $n_{K,C}$ .** By Norton [21],  $N_{F,-1} \ll_\epsilon |d_F|^{\frac{1}{4\sqrt{e}}+\epsilon} \ll_\epsilon |d_K|^{\frac{1}{4\sqrt{e}}+\epsilon}$ . Since  $n_{F,1} \leq N_{F,-1}$ ,

$$n_{K,C} \ll_\epsilon |d_F|^{\frac{1}{4\sqrt{e}}+\epsilon} \ll_\epsilon |d_K|^{\frac{1}{4\sqrt{e}}+\epsilon} \text{ for } C \subset A_n.$$

By Pollack [22],  $N_{F,1} \ll_\epsilon |d_F|^{\frac{1}{4}+\epsilon} \ll_\epsilon |d_K|^{\frac{1}{4}+\epsilon}$ . Since  $n_{F,-1} \leq N_{F,1}$ ,

$$n_{K,C} \ll_\epsilon |d_F|^{\frac{1}{4}+\epsilon} \ll_\epsilon |d_K|^{\frac{1}{4}+\epsilon} \text{ for } C \not\subset A_n.$$

**3.2. Conditional bounds of  $n_{K,C}$ .** We obtain conditional bounds on  $n_{F,C}$  under the zero-free region of  $L(s, \chi_F)$ , where  $\chi_F(p) = (\frac{d_F}{p})$ . Suppose  $L(s, \chi_F)$  is zero free on  $[\alpha, 1] \times [-(\log |d_F|)^2, (\log |d_F|)^2]$ . Then by [7],

$$-\frac{L'}{L}(\sigma, \chi_F) = \sum_{p < (\log |d_F|)^{16/(1-\alpha)}} \frac{\chi_F(p) \log p}{p^\sigma} + O(1),$$

for  $1 \leq \sigma \leq 3/2$ . Hence, it implies that

$$\left| -\frac{L'}{L}(\sigma, \chi_F) \right| \leq \frac{16}{(1-\alpha)} \log \log |d_F| + O(1).$$

Now for  $C \subset A_n$ , consider  $\zeta_F(s) = \zeta(s)L(s, \chi_F)$ . We obtain

$$\sum_p \frac{(1 + \chi_F(p)) \log p}{p^\sigma} = \frac{1}{\sigma - 1} - \frac{L'}{L}(\sigma, \chi_F) + O(1).$$

For a while, we assume that  $n_{K,C} \geq 3$ . For each prime  $p < n_{K,C}$ , the prime  $p$  splits in the quadratic resolvent  $F$ . (i.e.,  $\chi_F(p) = 1$  for all  $p < n_{K,C}$ .)

Then

$$\sum_{p < n_{K,C}} \frac{2 \log p}{p^\sigma} \leq \frac{1}{\sigma - 1} - \frac{L'}{L}(\sigma, \chi_F) + O(1).$$

By taking  $\sigma - 1 = \frac{\lambda}{\log n_{K,C}}$ , we have

$$\frac{1 - 2e^{-\lambda}}{2\lambda} \log n_{K,C} \leq \frac{16}{(1-\alpha)} \log \log |d_F| + O(1).$$

Hence

$$n_{K,C} \ll (\log |d_F|)^{\frac{16}{(1-\alpha)A}} \ll (\log |d_K|)^{\frac{16}{(1-\alpha)A}},$$

where  $A = \sup_{\lambda \geq 0} \frac{1-2e^{-\lambda}}{\lambda}$ , which is 0.37... when  $\lambda = 1.678...$  When  $n_{K,C} = 2$ , it clearly satisfies the above inequality. Hence we can remove the assumption  $n_{K,C} \geq 3$ .

Now for  $C \not\subset A_n$ , consider

$$\sum_p \frac{(1 - \chi_F(p)) \log p}{p^\sigma} = \frac{1}{\sigma - 1} + \frac{L'}{L}(\sigma, \chi_F) + O(1).$$

For each prime  $p < n_{K,C}$ , the prime  $p$  is inert in  $F$ . (i.e.,  $\chi_F(p) = -1$ ) for all  $p < n_{K,C}$ . We have the inequality

$$\sum_{p < n_{K,C}} \frac{2 \log p}{p^\sigma} \leq \frac{1}{\sigma - 1} + \frac{L'}{L}(\sigma, \chi_F) + O(1).$$

Hence

$$n_{K,C} \ll (\log |d_F|)^{\frac{16}{(1-\alpha)A}} \ll (\log |d_K|)^{\frac{16}{(1-\alpha)A}}.$$

(Here we implicitly assume that  $n_{K,C} \geq 3$  and remove the restriction after we obtain the upper bound.)

Let

$$L(X)^\pm = \{F \mid F : \text{quadratic field}, \pm d_F \leq X\}.$$

We may treat  $L(X)^\pm$  as families of quadratic Dirichlet  $L$ -functions  $L(s, \chi_F)$ . By applying Kowalski-Michel's theorem [16] to  $L(X)^\pm$ , we can show that every  $L$ -function in  $L(X)^\pm$  is zero-free on  $[\alpha, 1] \times [-(\log X)^2, (\log X)^2]$  except for  $O(X^{\beta_n/2})$   $L$ -functions. Here  $\beta_n$  is some small constant which appears in Conjecture 4.1. Here  $\alpha$  is a fixed constant close to 1 depending on  $\beta_n$  but independent of  $X$ . For the detail of how to apply Kowalski-Michel's theorem, we refer to [7].

#### 4. NUMBER FIELDS WITH THE SAME QUADRATIC RESOLVENT

Let  $F$  be a quadratic field. Then there are infinitely many  $S_n$ -fields having the same quadratic resolvent  $F$ . Let  $QR_n(X, F)$  be the set of  $S_n$ -fields with the common quadratic resolvent  $F$  and the absolute value of the discriminant bounded by  $X$ .

Since  $d_K = d_F m^2$  for some  $m \in \mathbb{Z}^+$ , by the counting conjecture (2.2), it is expected that  $|QR_n(X, F)| \ll (X/|d_F|)^{1/2+\epsilon}$ .

For our purpose, a weaker form is enough.

**Conjecture 4.1.** *There is a constant  $\beta_n$  with  $0 < \beta_n < 1/2$  for which*

$$|QR_n(X, F)| \ll \left( \frac{X}{|d_F|} \right)^{1-\beta_n},$$

where the implied constant is independent of  $F$ .

We prove it for  $n = 3$ . When  $n = 4$ , we are not able to prove it. However, in the Appendix, we obtain a good bound on the number of  $S_4$ -fields with the given cubic resolvent.

Given a quadratic field  $F$ , let

$$\Phi_F(s) = \frac{1}{2} + \sum_{K \in \mathcal{F}(F)} \frac{1}{f(K)^s},$$

where  $d_K = d_F f(K)^2$ , and  $\mathcal{F}(F)$  is the set of all cubic fields  $K$  with the quadratic resolvent field  $F$ .

Cohen and Thorne [9] found an explicit expression for  $\Phi_F(s)$ . Let  $D$  be a fundamental discriminant, and  $D^* = -3D$  if  $3 \nmid D$ , and  $D^* = -\frac{D}{3}$  if  $3 \mid D$ . Define  $\mathcal{L}_3(D) = \mathcal{L}_{D^*} \cup \mathcal{L}_{-27D}$ , where  $\mathcal{L}_N$  is the set of cubic fields of discriminant  $N$ .

By Theorem 2.5 in [9],

$$\Phi_F(s) = \sum_{i=1}^{|\mathcal{L}_3(d_F)+1|} \Phi_i(s), \quad \Phi_i(s) = \sum_{n=1}^{\infty} \frac{a_i(n)}{n^s},$$

and  $a_i(n) \leq 2^{\omega(n)} \ll 2^{\frac{\log n}{\log \log n}} \ll n^\epsilon$ . Hence each  $\Phi_i(s)$  is absolutely convergent for  $\operatorname{Re}(s) > 1$ . Also Theorem 2.5 in [9] implies

$$\Phi_i(1+c+it) \ll \left( \frac{\zeta(1+c)}{\zeta(2+2c)} \right)^2 \ll \frac{1}{c^2}.$$

Now we apply Perron's formula to each  $\Phi_i(s)$ . For  $c = \frac{1}{\log x}$ ,

$$\sum_{n < x} a_i(n) = \int_{1+c-ix}^{1+c+ix} \Phi_i(1+c+it) \frac{x^s}{s} ds + O(x^\epsilon),$$

with an absolute implied constant. Since  $\Phi_i(1+c+it) \ll (\log x)^2$ , the integral is majorized by  $x(\log x)^3$ . So

$$\sum_{n < x} a_i(n) \ll x(\log x)^3.$$

By [12],  $|\mathcal{L}_N| \ll |N|^{\frac{1}{3}+\epsilon}$ . Hence

$$|\{K \in \mathcal{F}(F) | f(K) \leq x\}| = \sum_{i=1}^{|\mathcal{L}_3(d_F)+1|} \sum_{n < x} a_i(n) \ll |d_F|^{\frac{1}{3}+\epsilon} x (\log x)^3.$$

Therefore,

$$|\{K \in \mathcal{F}(F) | f(K) \leq x\}| \ll x (\log x)^3 |d_F|^{\frac{1}{3}+\epsilon}.$$

Hence we have proved

**Proposition 4.2.**

$$|QR_3(X, F)| \ll X^{\frac{1}{2}} (\log X)^3 |d_F|^{-\frac{1}{6}+\epsilon},$$

with an absolute implied constant.

## 5. AVERAGE VALUE OF $n_{K,C}$

In this section, we prove Theorem 1.1 under Conjecture 4.1, and also under the counting conjectures (2.2) – (2.3).

For simplicity of notation, we denote  $L_n^{(r_2)}(X)$  by  $L(X)$ . Take  $y = \frac{1-\delta}{4\gamma} \log X$ , where  $\delta$  and  $\gamma$  are the constants in (2.2) and (2.3). Then

$$\sum_{K \in L(X)} n_{K,C} = \sum_{K \in L(X), n_{K,C} \leq y} n_{K,C} + \sum_{K \in L(X), n_{K,C} > y} n_{K,C}.$$

Here  $n_{K,C} = q$  means that for all primes  $p < q$ ,  $\text{Frob}_p \in C$  and  $q$  is ramified or  $\text{Frob}_q \notin C$ . By the counting conjectures, there are

$$\frac{1 - |C|/|S_n| + f(q)}{1 + f(q)} \prod_{p < q} \frac{|C|/|S_n|}{1 + f(p)} A(r_2)X + O(X^{\frac{1+3\delta}{4}})$$

such number fields in  $L(X)$ . Hence,

$$\begin{aligned} \sum_{K \in L(X), n_{K,C} \leq y} n_{K,C} &= \sum_{q \leq y} q \sum_{K \in L(X), n_{K,C} = q} 1 \\ &= A(r_2)X \sum_{q \leq y} \frac{q(1 - |C|/|S_n| + f(q))}{1 + f(q)} \prod_{p < q} \frac{|C|/|S_n|}{1 + f(p)} + O(y^2 X^{\frac{1+3\delta}{4}}) \\ &= A(r_2)X \sum_q \frac{q(1 - |C|/|S_n| + f(q))}{1 + f(q)} \prod_{p < q} \frac{|C|/|S_n|}{1 + f(p)} \\ &\quad + O\left(X \sum_{q > y} q \prod_{p < q} |C|/|S_n| + (\log X)^2 X^{\frac{1+3\delta}{4}}\right). \end{aligned}$$



Since

$$\sum_{q>y} q \prod_{p<q} |C|/|S_n| \leq \sum_{q>y} q \left( \frac{|C|}{|S_n|} \right)^{\pi(q)} \ll \frac{1}{\log X},$$

$$\sum_{K \in L(X), n_{K,C} \leq y} n_{K,C} = A(r_2) X \sum_q \frac{q(1 - |C|/|S_n| + f(q))}{1 + f(q)} \prod_{p<q} \frac{|C|/|S_n|}{1 + f(p)} + O\left(\frac{X}{\log X}\right).$$

Now we divide the sum  $\sum_{|d_K| \leq X, n_{K,C} > y} n_{K,C}$  into two subsums. Let  $E(X)$  be the set of  $S_n$ -fields in  $L(X)$  for which the quadratic  $L$ -function  $L(s, \chi_F)$ , where  $F$  is the quadratic resolvent, may not have the desired zero-free region in Section 3.2.

$$(5.1) \quad \sum_{K \in L(X), n_{K,C} > y} n_{K,C} = \sum_{n_{K,C} > y, K \notin E(X)} n_{K,C} + \sum_{n_{K,C} > y, K \in E(X)} n_{K,C}.$$

Let's deal with the second sum. From the unconditional bound in Section 3.1,  $n_{K,C} \ll |d_F|^{\frac{1}{4\sqrt{\epsilon}} + \epsilon}$  or  $|d_F|^{\frac{1}{4} + \epsilon}$  depending on whether  $C \subset A_n$  or  $C \not\subset A_n$ . In any case  $n_{K,C} \ll |d_F|^{\frac{1}{4} + \epsilon}$ . For such  $F$ , by Conjecture 4.1, we have at most  $(X/|d_F|)^{1-\beta_n}$   $S_n$ -fields with the same quadratic resolvent  $F$ . Hence such  $F$  contributes at most  $|d_F|^{\frac{1}{4} + \epsilon} (X/|d_F|)^{1-\beta_n} \ll X^{1-\beta_n}$ . In Section 3.2, we showed that there are at most  $X^{\beta_n/2}$  quadratic fields  $F$  which may not have the desired zero-free region. Therefore,

$$\sum_{n_{K,C} > y, K \in E(X)} n_{K,C} \ll X^{\beta_n/2} X^{1-\beta_n} \ll X^{1-\beta_n/2}.$$

To handle the first sum, we use the conditional bound on  $n_{K,C}$  in Section 3.2.

$$\begin{aligned} \sum_{n_{K,C} > y, K \notin E(X)} n_{K,C} &\ll (\log X)^{\frac{16}{(1-\alpha)A}} \sum_{n_{K,C} > y} 1 \\ &\ll (\log X)^{\frac{16}{(1-\alpha)A}} \left( X \prod_{p<y} |C|/|S_n| + X^\delta \left( \prod_{p<y} p \right)^\gamma \right) \\ &\ll X (\log X)^{\frac{16}{(1-\alpha)A}} \left( \frac{|C|}{|S_n|} \right)^{\pi(y)} + X^{\frac{1+\delta}{2}}. \end{aligned}$$

We use the fact that  $\left( \frac{|C|}{|S_n|} \right)^{\pi(y)} \ll e^{-\frac{\log X}{\log \log X}} \ll (\log X)^{-k}$  for any  $k$ . Our discussion is summarized as follows:

**Theorem 5.2.** *Let  $L_n^{(r_2)}(X)$  be the set of  $S_n$ -fields  $K$  of signature  $(r_1, r_2)$  with  $|d_K| < X$ . Assume the counting conjectures (2.2) – (2.3) and Conjecture 4.1. Let  $C$  be a conjugacy class of  $S_n$  and*

$n_{K,C}$  be the least prime with  $\text{Frob}_p \notin C$ . Then,

$$(5.3) \quad \frac{1}{|L_n^{(r_2)}(X)|} \sum_{K \in L_n^{(r_2)}(X)} n_{K,C} = \sum_q \frac{q(1 - |C|/|S_n| + f(q))}{1 + f(q)} \prod_{p < q} \frac{|C|/|S_n|}{1 + f(p)} + O\left(\frac{1}{\log X}\right).$$

For  $S_3$ -fields, the counting conjectures (2.2) – (2.3) and Conjecture 4.1 are true. Hence, the above theorem holds unconditionally. For  $S_4$  and  $S_5$ -fields, the counting conjectures (2.2) – (2.3) are true. Hence under Conjecture 4.1, Theorem 5.2 holds for  $S_4$  and  $S_5$ -fields.

The tables below show average values of  $n_{K,C}$  for  $S_3$ ,  $S_4$  and  $S_5$ -fields. The computations are done by PARI.

$S_3$	Average of $n_{K,C}$	$S_4$	Average of $n_{K,C}$	$S_5$	Average of $n_{K,C}$
$[e]$	2.1211027...	$[e]$	2.0206694...	$[e]$	2.0036404...
$[(12)]$	2.6719625...	$[(12)(34)]$	2.0691556...	$[(12)(34)]$	2.0632551...
$[(123)]$	2.3192802...	$[(1234)]$	2.1653006...	$[(123)]$	2.0891619...
		$[(12)]$	2.1653006...	$[(12)(345)]$	2.0891619...
		$[(123)]$	2.2516575...	$[(12)]$	2.0399630...
				$[(1234)]$	2.1505010...
				$[(12345)]$	2.1120340...

**Remark 5.4.** From the tables above, we can see that the average value of  $n_{K,C}$  is close to 2 and  $n_{K,C} < n_{K,C'}$  if  $|C| < |C'|$ . In fact, it is expected from the formula for the average value of  $n_{K,C}$ . The probability for  $n_{K,C}$  to be 2 is  $\frac{1 - |C|/|S_n| + f(2)}{1 + f(2)}$ , which happens to most of the number fields. For example, for  $S_5$ -fields, the probability for  $n_{K,[e]}$  to be 2 is 0.996396...

**Remark 5.5.** Let  $L(X)^\pm$  be the set of real/complex quadratic extension  $F$  with  $\pm d_F \leq X$ . For the sake of completeness, we record the average of  $n_{F,\pm 1}$ . It is easy to check that the probabilities for a prime  $p$  is to ramify, split, or be inert are  $\frac{1}{p+1}$ ,  $\frac{p}{2(p+1)}$ , or  $\frac{p}{2(p+1)}$  respectively. Hence,

$$\lim_{X \rightarrow \infty} \frac{\sum_{\pm d_F \leq X} n_{F,\pm 1}}{|L(X)^\pm|} = \sum_q \frac{q^2 + 2q}{2(q+1)} \prod_{p < q} \frac{p}{2(p+1)} = 2.83264 \dots$$

## 6. ALTERNATIVE PROOF OF (5.3) WITHOUT CONJECTURE 4.1

In this section, we show how we can avoid using Conjecture 4.1 which is necessary to estimate (5.1). Instead we assume the strong Artin conjecture and use various Artin  $L$ -functions, depending on the conjugacy class. We consider  $S_n$ -fields for  $n = 3, 4, 5$ . Since the strong Artin

conjecture is known for  $S_3$  and  $S_4$ -fields [5], our result is unconditional for  $S_3$  and  $S_4$ -fields. We show, by a case by case analysis on each  $C$ ,

$$\sum_{K \in L(X), n_{K,C} > y} n_{K,C} = O\left(\frac{X}{\log X}\right).$$

We still divide it into two subsums

$$\sum_{K \in L(X), n_{K,C} > y} n_{K,C} = \sum_{n_{K,C} > y, K \notin E(X)} n_{K,C} + \sum_{n_{K,C} > y, K \in E(X)} n_{K,C}.$$

However, the exceptional set  $E(X)$  will be different for each  $C$ , since we consider zero-free regions of different Artin  $L$ -functions. The second sum is estimated by using the unconditional bounds of  $n_{K,C}$  in Section 3.1. For the first sum, we need conditional bounds, conditional on zero-free regions of various Artin  $L$ -functions. We use the following formula as in [18]: For a conjugacy class  $C$  of  $S_n$ , define, for  $\sigma > 1$ ,

$$(6.1) \quad F_C(\sigma) = -\frac{|C|}{|S_n|} \sum_{\psi} \bar{\psi}(C) \frac{L'}{L}(\sigma, \psi, \widehat{K}/\mathbb{Q}),$$

where  $\psi$  runs over the irreducible characters of  $S_n$  and  $L(s, \psi, \widehat{K}/\mathbb{Q})$  is the Artin  $L$ -function attached to the character  $\psi$ . By orthogonality of characters,

$$(6.2) \quad F_C(\sigma) = \sum_p \sum_{m=1}^{\infty} \frac{\theta(p^m) \log p}{p^{m\sigma}},$$

where for a prime  $p$  unramified in  $\widehat{K}$ ,

$$\theta(p^m) = \begin{cases} 1 & \text{if } (\text{Frob}_p)^m \in C, \\ 0 & \text{otherwise.} \end{cases}$$

and  $0 \leq \theta(p^m) \leq 1$  if  $p$  ramifies in  $\widehat{K}$ .

**6.1.  $S_4$ -fields.** Here, we follow the notations in [11] for characters of  $S_4$ .

**6.1.1. Case 1.  $C = (1234)$ .** From (6.2),

$$\sum_{p \in C} \frac{\log p}{p^\sigma} = \frac{1}{4} \cdot \frac{1}{\sigma - 1} - \frac{1}{4} \left( -\frac{L'}{L}(s, \chi_2) - \frac{L'}{L}(\sigma, \chi_4) + \frac{L'}{L}(\sigma, \chi_5) \right) + O(1).$$

Here

$$-\frac{L'}{L}(\sigma, \chi_2) - \frac{L'}{L}(\sigma, \chi_4) = \sum_p \frac{\chi_2(p) + \chi_4(p)}{p^\sigma} + O(1) \geq -2 \sum_{p \in C} \frac{\log p}{p^\sigma} + O(1).$$

Hence we have

$$(6.3) \quad \frac{1}{2} \sum_{p \in C} \frac{\log p}{p^\sigma} \leq \frac{1}{4} \cdot \frac{1}{\sigma-1} - \frac{1}{4} \frac{L'}{L}(\sigma, \chi_5) + O(1).$$

Since  $\chi_5 = \chi_4 \otimes \chi_2$ , the conductor of  $\chi_5$  is at most  $|d_K|^3$ . Since  $\chi_4$  is modular [5],  $\chi_5$  is modular, i.e.,  $L(s, \chi_5)$  is a cuspidal automorphic  $L$ -function of  $GL_3/\mathbb{Q}$ . Consider a family of Artin  $L$ -functions:

$$L(X) = \{L(s, \chi_5) \mid K \in L(X)^{r_2}\}.$$

Then all  $L$ -functions in  $L(X)$  are distinct because  $L(s, \chi_4)$ 's are distinct since  $K$  is arithmetically solitary. (See [7] for the detail.) By applying Kowalski-Michel's theorem to  $L(X)$ , every  $L$ -function in the family is zero-free on  $[\alpha, 1] \times [-(3 \log X)^2, (3 \log X)^2]$  except for  $O(X^{1/1000})$   $L$ -functions.

For a  $L$ -function with such a zero-free region,

$$(6.4) \quad -\frac{L'}{L}(\sigma, \chi_5) \leq \frac{16 \cdot 3}{(1-\alpha)} \log \log d_K + O(1),$$

for  $1 \leq \sigma \leq 3/2$ . (See (5.1) in [7]). Plugging (6.4) into (6.3), and taking  $\sigma = 1 + \frac{\lambda}{\log n_{K,C}}$ , we obtain  $\frac{1-2e^{-\lambda}}{4\lambda} \log n_{K,C} \leq \frac{12}{(1-\alpha)} \log \log |d_K| + O(1)$ . Hence,

$$(6.5) \quad n_{K,C} \ll (\log |d_K|)^{\frac{48}{(1-\alpha)A}},$$

where  $A = \sup_{\lambda \geq 0} \frac{1-2e^{-\lambda}}{\lambda}$ .

Let  $E(X)$  be the exceptional subset in  $L(X)$ . Then  $|E(X)| \ll X^{1/1000}$ . By abuse of language,  $K \notin E(X)$  means  $L(s, \chi_5) \notin E(X)$ . Then

$$\begin{aligned} \sum_{K \in L(X), n_{K,C} > y} n_{K,C} &= \sum_{K \notin E(X), n_{K,C} > y} n_{K,C} + \sum_{K \in E(X), n_{K,C} > y} n_{K,C} \\ &\ll (\log X)^{\frac{48}{(1-\alpha)A}} \sum_{K \in L(X)^{r_2}, n_{K,C} > y} 1 + X^{1/4+\epsilon} \cdot X^{1/1000} \\ &\ll (\log X)^{\frac{48}{(1-\alpha)A}} \left( X \left( \frac{|C|}{|S_4|} \right)^{\pi(y)} + X^{\frac{1+\delta}{2}} \right) \ll \frac{X}{\log X}. \end{aligned}$$

6.1.2. *Case 2.*  $C = (12)(34)$ . From (6.2)

$$\sum_{p \in C} \frac{\log p}{p^\sigma} = \frac{1}{8} \cdot \frac{1}{\sigma-1} - \frac{1}{8} \left( \frac{L'}{L}(\sigma, \chi_2) + 2 \frac{L'}{L}(\sigma, \chi_3) - \frac{L'}{L}(\sigma, \chi_4) - \frac{L'}{L}(\sigma, \chi_5) \right) + O(1).$$

Since

$$-\frac{L'}{L}(\sigma, \chi_2) - 2\frac{L'}{L}(\sigma, \chi_3) \leq 5 \sum_p \frac{\log p}{p^\sigma} + O(1) = \frac{5}{\sigma-1} + O(1),$$

we have

$$\sum_{p \in C} \frac{\log p}{p^\sigma} \leq \frac{6}{8} \cdot \frac{1}{\sigma-1} + \frac{1}{8} \left( \frac{L'}{L}(\sigma, \chi_4) + \frac{L'}{L}(\sigma, \chi_5) \right) + O(1).$$

Now consider a family of Artin  $L$ -functions:

$$L(X) = \{L(s, \chi_4)L(s, \chi_5) \mid K \in L(X)^{r_2}\}.$$

We apply Kowalski-Michel's theorem to  $L(X)$ . Then every  $L$ -function in  $L(X)$  is zero-free on  $[\alpha, 1] \times [-(4 \log |d_K|)^2, (4 \log |d_K|)^2]$  except for  $O(X^{1/1000})$   $L$ -functions. Since  $L(s, \chi_4)$  and  $L(s, \chi_5)$  are simultaneously zero-free on  $[\alpha, 1] \times [-(4 \log |d_K|)^2, (4 \log |d_K|)^2]$ ,

$$\left| -\frac{L'}{L}(\sigma, \chi_4) \right|, \quad \left| -\frac{L'}{L}(\sigma, \chi_5) \right| \leq \frac{16 \cdot 3}{(1-\alpha)} \log \log |d_K| + O(1),$$

and with this conditional bound, as we did in the previous section, we can show

$$\sum_{K \in L(X), n_{K,C} > y} n_{K,C} = O\left(\frac{X}{\log X}\right).$$

6.1.3. *Case 3.*  $C = (12)$ . From (6.2),

$$\sum_{p \in C} \frac{\log p}{p^\sigma} = \frac{1}{4} \cdot \frac{1}{\sigma-1} - \frac{1}{4} \left( -\frac{L'}{L}(\sigma, \chi_2) + \frac{L'}{L}(\sigma, \chi_4) - \frac{L'}{L}(\sigma, \chi_5) \right) + O(1).$$

Here

$$-\frac{L'}{L}(\sigma, \chi_2) - \frac{L'}{L}(\sigma, \chi_5) \geq -2 \sum_{p \in C} \frac{\log p}{p^\sigma} + O(1).$$

Hence we have

$$\frac{1}{2} \sum_{p \in C} \frac{\log p}{p^\sigma} \leq \frac{1}{4} \cdot \frac{1}{\sigma-1} - \frac{1}{4} \cdot \frac{L'}{L}(\sigma, \chi_4) + O(1).$$

Let  $\tilde{L}(X) = \{L(s, \chi_4) \mid K \in L(X)\}$  and apply Kowalski-Michel's theorem to  $\tilde{L}(X)$ . Then every  $L(s, \chi_4)$  in  $\tilde{L}(X)$  except for  $O(X^{1/1000})$   $L$ -functions satisfies

$$\left| -\frac{L'}{L}(\sigma, \chi_4) \right| \leq \frac{16 \cdot 3}{(1-\alpha)} \log \log |d_K| + O(1),$$

and again we have

$$\sum_{K \in L(X), n_{K,C} > y} n_{K,C} = O\left(\frac{X}{\log X}\right).$$

6.1.4. *Case 4.*  $C = (123)$ . From (6.2),

$$\sum_{p \in C} \frac{\log p}{p^\sigma} = \frac{1}{3} \cdot \frac{1}{\sigma-1} - \frac{1}{3} \left( \frac{L'}{L}(\sigma, \chi_2) - \frac{L'}{L}(\sigma, \chi_3) \right) + O(1).$$

Since

$$-\frac{L'}{L}(\sigma, \chi_2) \leq \frac{1}{\sigma-1} + O(1),$$

we have,

$$\sum_{p \in C} \frac{\log p}{p^\sigma} \leq \frac{2}{3} \cdot \frac{1}{\sigma-1} + \frac{1}{3} \cdot \frac{L'}{L}(\sigma, \chi_3) + O(1).$$

Here  $L(s, \chi_3) = \frac{\zeta_M(s)}{\zeta(s)}$ , where  $M$  is the cubic resolvent of  $K$ . Note that  $M$  is an  $S_3$ -field. Let

$$\tilde{L}(X) = \{L(s, \chi_3) \mid M: S_3\text{-field}, |d_M| \leq X\}.$$

By applying Kowalski-Michel's theorem to  $\tilde{L}(X)$ , we can see that every  $L(s, \chi_3)$  in  $\tilde{L}(X)$  except for  $O(X^{1/1000})$   $L$ -functions satisfies

$$\left| -\frac{L'}{L}(\sigma, \chi_3) \right| \leq \frac{16 \cdot 3}{(1-\alpha)} \log \log |d_K| + O(1),$$

and with this bound, we have  $n_{K,C} \ll (\log |d_K|)^{\frac{16}{(1-\alpha)A}}$ , where  $A = \sup_{\lambda \geq 0} \frac{1-3e^{-\lambda}}{3\lambda} = 0.10\dots$  when  $\lambda = 2.29\dots$

Let  $E(X) = \{K \in L(X) \mid L(s, \chi_3) \text{ belongs to the exceptional subset in } \tilde{L}(X)\}$ . Note that there are at most  $X^{\frac{1}{2}+\epsilon}$   $S_4$ -fields in  $L(X)$  which have the common cubic resolvent  $M$ . (See the Appendix: Section 9.) Hence  $|E(X)| \ll X^{1/1000} X^{\frac{1}{2}+\epsilon}$ . Then

$$\begin{aligned} \sum_{K \in L(X), n_{K,C} > y} n_{K,C} &= \sum_{K \notin E(X), n_{K,C} > y} n_{K,C} + \sum_{K \in E(X), n_{K,C} > y} n_{K,C} \\ &\ll (\log X)^{\frac{16}{(1-\alpha)A}} \sum_{K \in L(X), n_{K,C} > y} 1 + X^{1/4+\epsilon} \cdot X^{1/1000} \cdot X^{\frac{1}{2}+\epsilon} = O\left(\frac{X}{\log X}\right). \end{aligned}$$

6.1.5. *Case 5.*  $C = e$ . In [7], we showed that if  $L(s, \chi_4) = \zeta_K(s)/\zeta(s)$  is entire and zero-free on  $[\alpha, 1] \times [-(\log |d_K|)^2, (\log |d_K|)^2]$ , then  $n_{K,e} \ll (\log |d_K|)^{\frac{16}{(1-\alpha)A}}$ , where  $A = \sup_{\lambda \geq 0} \frac{1-\frac{4}{3}e^{-\lambda}}{\lambda} = 0.5\dots$  when  $\lambda = 0.96\dots$  [There is a typo in [7], Theorem 1.1.] Since every field  $K$  in  $L(X)$  except for  $O(X^{1/1000})$  fields has such upper bound,

$$\sum_{K \in L(X)^{r_2}, n_{K,C} > y} n_{K,C} = O\left(\frac{X}{\log X}\right).$$

**6.2.  $S_5$ -fields.** We assume the strong Artin conjecture for  $S_5$  fields, and follow notations in [11] for characters of  $S_5$ . Then  $L(s, \chi_3) = \zeta_K(s)/\zeta(s)$ , and  $L(s, \chi_5) = \zeta_H(s)/\zeta(s)$ , where  $H$  is the sextic resolvent of  $K$ . For the sign character  $\chi_2$ ,

$$\chi_4 = \chi_3 \otimes \chi_2, \quad \chi_6 = \chi_5 \otimes \chi_2, \quad \text{and} \quad \chi_7 = \wedge^2 \chi_3$$

**6.2.1. Case 1.**  $C = (12345)$ . From (6.2),

$$\sum_{p \in C} \frac{\log p}{p^\sigma} = \frac{1}{5} \cdot \frac{1}{\sigma-1} - \frac{1}{5} \left( \frac{L'}{L}(\sigma, \chi_2) - \frac{L'}{L}(\sigma, \chi_3) - \frac{L'}{L}(\sigma, \chi_4) + \frac{L'}{L}(\sigma, \chi_7) \right) + O(1).$$

Since

$$\frac{L'}{L}(\sigma, \chi_3) + \frac{L'}{L}(\sigma, \chi_4) \leq 2 \sum_{p \in C} \frac{\log p}{p^\sigma} + O(1), \quad -\frac{L'}{L}(\sigma, \chi_2) \leq \frac{1}{\sigma-1} + O(1),$$

we have

$$\frac{3}{5} \sum_{p \in C} \frac{\log p}{p^\sigma} \leq \frac{2}{5} \cdot \frac{1}{\sigma-1} - \frac{1}{5} \cdot \frac{L'}{L}(\sigma, \chi_7) + O(1).$$

Define  $\tilde{L}(X) = \{L(s, \chi_7) \mid K \in L(X)\}$ . Note that the conductor of  $L(s, \chi_7)$  is bounded by  $|d_K|^7$  [2]. (G. Henniart noted in a private communication that it can be improved to  $|d_K|^{\frac{3}{2}}$ .)

**Lemma 6.6.** *Let  $L(s, \chi_7) = L(s, \chi_7, \widehat{K}/\mathbb{Q})$  and  $L(s, \chi'_7) = L(s, \chi'_7, \widehat{K'}/\mathbb{Q})$ . Suppose  $L(s, \chi_7) = L(s, \chi'_7)$ . Then  $K$  and  $K'$  are conjugate.*

*Proof.* Recall the following from [4]:  $\chi_3 = As(\sigma)$ , the Asai lift of  $\sigma$ , which is a 2-dimensional representation of  $\tilde{A}_5$  over  $F = \mathbb{Q}[\sqrt{d_K}]$ . Let  $\pi$  be the cuspidal representation of  $GL_2/F$  corresponding to  $\sigma$ , and let  $\Pi$  be the cuspidal representation of  $GL_4/\mathbb{Q}$  corresponding to  $\chi_3$  by the strong Artin conjecture. Then  $\wedge^2 \Pi \simeq I_F^\mathbb{Q}(Sym^2 \pi)$ .

Let  $\pi', \Pi'$  be defined by  $K'$ . Suppose  $L(s, \Pi, \wedge^2) = L(s, \Pi', \wedge^2)$ . Then  $L(s, Sym^2(\pi)) = L(s, Sym^2(\pi'))$ . By Ramakrishnan [25],  $\pi' \simeq \pi \otimes \chi$  for a quadratic character of  $F$ . Then by Krishnamurthy [17],  $As(\pi) \simeq As(\pi')$ . Hence  $\Pi \simeq \Pi'$ . Therefore  $L(s, \chi_3, K/\mathbb{Q}) = L(s, \chi_3, K'/\mathbb{Q})$ . Since  $K$  is arithmetically solitary,  $K$  and  $K'$  are conjugate.  $\square$

Hence  $L$ -functions in  $\tilde{L}(X)$  are all distinct. By applying Kowalski-Michel's theorem to  $\tilde{L}(X)$ , we can show that every  $L(s, \chi_7)$  in  $\tilde{L}(X)$  except for  $O(X^{1/1000})$   $L$ -function satisfies

$$\left| -\frac{L'}{L}(\sigma, \chi_7, \widehat{K}/\mathbb{Q}) \right| \leq \frac{16 \cdot 28}{(1-\alpha)} \log \log |d_K| + O(1).$$

Hence

$$\sum_{K \in L(X), n_{K,C} > y} n_{K,C} = O\left(\frac{X}{\log X}\right).$$

6.2.2. *Case 2.*  $C = (1234)$ . From (6.2),

$$\sum_{p \in C} \frac{\log p}{p^\sigma} = \frac{1}{4} \cdot \frac{1}{\sigma-1} - \frac{1}{4} \left( -\frac{L'}{L}(\sigma, \chi_2) + \frac{L'}{L}(\sigma, \chi_5) - \frac{L'}{L}(\sigma, \chi_6) \right) + O(1).$$

Since

$$\frac{L'}{L}(\sigma, \chi_2) + \frac{L'}{L}(\sigma, \chi_6) \leq 2 \sum_{p \in C} \frac{\log p}{p^\sigma},$$

we have

$$\frac{1}{2} \sum_{p \in C} \frac{\log p}{p^\sigma} \leq \frac{1}{4} \cdot \frac{1}{\sigma-1} - \frac{1}{4} \cdot \frac{L'}{L}(\sigma, \chi_5).$$

**Lemma 6.7.** *Let  $L(s, \chi_5) = L(s, \chi_5, \widehat{K}/\mathbb{Q})$  and  $L(s, \chi'_5) = L(s, \chi'_5, \widehat{K'}/\mathbb{Q})$ . Suppose  $L(s, \chi_5) = L(s, \chi'_5)$ . Then  $K$  and  $K'$  are conjugate.*

*Proof.* It is easy to see (cf. [14], page 28)  $\wedge^2 \chi_5 = \chi_3 \otimes \chi_2 \oplus \wedge^2 \chi_3$ . Now let  $\chi'_3, \chi'_5$  be defined by  $K'$ , and suppose  $\chi_5 \simeq \chi'_5$ . Then  $\wedge^2 \chi_5 \simeq \wedge^2 \chi'_5$ . Hence  $\chi_3 \otimes \chi_2 \oplus \wedge^2 \chi_3 \simeq \chi'_3 \otimes \chi'_2 \oplus \wedge^2 \chi'_3$ . By strong multiplicity one,  $\wedge^2 \chi_3 \simeq \wedge^2 \chi'_3$ . Hence by Lemma 6.6,  $\chi_3 \simeq \chi'_3$ .  $\square$

Hence by applying Kowalski-Michel to  $\tilde{L}(X) = \{L(s, \chi_5) \mid K \in L(X)\}$ , we proceed as in Case 1.

6.2.3. *Case 3.*  $C = (12)(345)$ . From (6.2),

$$\sum_{p \in C} \frac{\log p}{p^\sigma} = \frac{1}{6} \cdot \frac{1}{\sigma-1} - \frac{1}{6} \left( -\frac{L'}{L}(\sigma, \chi_2) - \frac{L'}{L}(\sigma, \chi_3) + \frac{L'}{L}(\sigma, \chi_4) - \frac{L'}{L}(\sigma, \chi_5) + \frac{L'}{L}(\sigma, \chi_6) \right) + O(1).$$

Since

$$\frac{L'}{L}(\sigma, \chi_2) + \frac{L'}{L}(\sigma, \chi_3) + \frac{L'}{L}(\sigma, \chi_5) \leq 3 \sum_{p \in C} \frac{\log p}{p^\sigma},$$

we have

$$\frac{1}{2} \sum_{p \in C} \frac{\log p}{p^\sigma} \leq \frac{1}{6} \cdot \frac{1}{\sigma-1} - \frac{1}{6} \cdot \frac{L'}{L}(\sigma, \chi_4) - \frac{1}{6} \cdot \frac{L'}{L}(\sigma, \chi_6).$$

Apply Kowalski-Michel to  $\tilde{L}(X) = \{L(s, \chi_4)L(s, \chi_6) \mid K \in L(X)\}$ , and we proceed as in Case 1.



6.2.4. *Case 4.*  $C = (12)(34)$ . From (6.2),

$$\sum_{p \in C} \frac{\log p}{p^\sigma} = \frac{1}{8} \cdot \frac{1}{\sigma-1} - \frac{1}{8} \left( \frac{L'}{L}(\sigma, \chi_2) + \frac{L'}{L}(\sigma, \chi_5) + \frac{L'}{L}(\sigma, \chi_6) - 2 \frac{L'}{L}(\sigma, \chi_7) \right) + O(1).$$

Since

$$\frac{L'}{L}(\sigma, \chi_7) \leq 2 \sum_{p \in C} \frac{\log p}{p^\sigma} + O(1), \quad -\frac{L'}{L}(\sigma, \chi_2) \leq \frac{1}{\sigma-1} + O(1),$$

we have

$$\frac{1}{2} \sum_{p \in C} \frac{\log p}{p^\sigma} \leq \frac{1}{4} \cdot \frac{1}{\sigma-1} - \frac{1}{8} \left( \frac{L'}{L}(\sigma, \chi_5) + \frac{L'}{L}(\sigma, \chi_6) \right).$$

Apply Kowalski-Michel to  $\tilde{L}(X) = \{L(s, \chi_5)L(s, \chi_6) \mid K \in L(X)\}$ , and we proceed as in Case 1.

6.2.5. *Case 5.*  $C = (12)$ . We use the  $L$ -function  $L(s, \chi_3)$ :

$$-\frac{L'}{L}(\sigma, \chi_3) = \sum_p \frac{\chi_3(p) \log p}{p^\sigma} + O(1).$$

Note that  $\chi_3(p) = 2$  if  $\text{Frob}_p \in C$ , and  $1 + \chi_3(p) \geq 1$ . Then,

$$3 \sum_{p < n_{K,C}} \frac{\log p}{p^\sigma} \leq -\frac{\zeta'}{\zeta}(\sigma) - \frac{L'}{L}(\sigma, \chi_3) + O(1).$$

By applying Kowalski-Michel to the set  $\tilde{L}(X) = \{L(s, \chi_3) \mid K \in L(X)\}$ , we obtain that every  $L(s, \chi_3)$  in  $\tilde{L}(X)$  except for  $O(X^{1/1000})$   $L$ -function satisfies

$$-\frac{L'}{L}(\sigma, \chi_3) \leq \frac{64}{1-\alpha} \log \log |d_K| + O(1).$$

Hence by taking  $\sigma - 1 = \frac{\lambda}{\log n_{K,C}}$ , we have  $n_{K,C} \leq (\log |d_K|)^{\frac{64}{(1-\alpha)A}}$ , where  $A = \sup_{\lambda > 0} \frac{2-3e^{-\lambda}}{\lambda}$ .

We obtain

$$\sum_{K \in L(X), n_{K,C} > y} n_{K,C} = O\left(\frac{X}{\log X}\right).$$

6.2.6. *Case 6.*  $C = (123)$ . Since  $\chi_3(p) = 1$  if  $\text{Frob}_p \in C$ , we can use  $L(s, \chi_3)$ . This case is similar to the case  $C = (12)$ .

6.2.7. *Case 7.*  $C = e$ . Since  $\chi_3(p) = 4$  if  $\text{Frob}_p \in C$ , we can use  $L(s, \chi_3)$ . This case is similar to the case  $C = (12)$ .

6.3.  **$S_3$ -fields.** For the sake of completeness, we include the case of  $S_3$ . Here, we follow the notations in [11] for characters of  $S_3$ .

6.3.1. *Case 1.*  $C = (123)$ . From (6.2),

$$\sum_{p \in C} \frac{\log p}{p^\sigma} = \frac{1}{3} \cdot \frac{1}{\sigma - 1} - \frac{1}{3} \left( \frac{L'}{L}(\sigma, \chi_2) - \frac{L'}{L}(\sigma, \chi_3) \right) + O(1).$$

Then

$$\sum_{p < n_{K,C}} \frac{\log p}{p^\sigma} \leq \frac{2}{3} \cdot \frac{1}{\sigma - 1} + \frac{1}{3} \cdot \frac{L'}{L}(\sigma, \chi_3) + O(1).$$

Since  $L(s, \chi_3) = \frac{\zeta_K(s)}{\zeta(s)}$ ,  $L(s, \chi_3)$  is modular, i.e.,  $L(s, \chi_3)$  is a cuspidal automorphic  $L$ -function of  $GL_2/\mathbb{Q}$ . This case is similar to  $S_4$ ,  $C = (1234)$ .

6.3.2. *Case 2.*  $C = (12)$ . From (6.2),

$$\sum_{p \in C} \frac{\log p}{p^\sigma} = \frac{1}{2} \cdot \frac{1}{\sigma - 1} + \frac{1}{2} \cdot \frac{L'}{L}(\sigma, \chi_2) + O(1).$$

This case was done in Section 3.2.

6.3.3. *Case 3.*  $C = e$ . This case is similar to  $S_4$ ,  $C = e$ .

We summarize it as

**Theorem 6.8.** *Theorem 1.1 holds unconditionally for  $S_3$  and  $S_4$ -fields. Under the strong Artin conjecture for  $S_5$ -fields, Theorem 1.1 holds.*

**Remark 6.9.** The above method can be generalized to  $S_n$ -fields and a special conjugacy class: Let  $K$  be an  $S_n$ -field, and let  $L(s, \chi) = \frac{\zeta_K(s)}{\zeta(s)}$ . Let  $C$  be a conjugacy class of  $S_n$  such that  $\chi(C) \geq 1$ . Then under the counting conjectures (2.2) – (2.3) and the strong Artin conjecture for  $L(s, \chi)$ , we have

$$\frac{1}{|L_n^{(r_2)}(X)|} \sum_{K \in L_n^{(r_2)}(X)} n_{K,C} = \sum_q \frac{q(1 - |C|/|S_n| + f(q))}{1 + f(q)} \prod_{p < q} \frac{|C|/|S_n|}{1 + f(p)} + O\left(\frac{1}{\log X}\right).$$

## 7. AVERAGE VALUE OF $N_{K,C}$

In this section, we compute the averages of  $N_{K,C}$ . We need to assume the GRH for Dedekind zeta functions because we do not have a good bound on  $N_{K,C}$ . We generalize Martin and Pollack's result for  $S_3$ -fields. (See Theorem 4.8 [20].) Since the idea of proof is similar to the case of  $n_{K,C}$ , we omit some details. Consider

$$\sum_{K \in L(X)} N_{K,C} = \sum_{K \in L(X), N_{K,C} \leq y} N_{K,C} + \sum_{K \in L(X), N_{K,C} > y} N_{K,C}.$$

Here  $N_{K,C} = q$  means that for all primes  $p < q$ ,  $\text{Frob}_p \notin C$  and  $\text{Frob}_q \in C$ . By the counting conjectures, there are

$$\frac{|C|/|S_n|}{1+f(q)} \prod_{p < q} \frac{1 - |C|/|S_n| + f(p)}{1+f(p)} A(r_2)X + O(X^{\frac{1+3\delta}{4}}).$$

such number fields in  $L(X)$ . Hence,

$$\begin{aligned} \sum_{K \in L(X), N_{K,C} \leq y} N_{K,C} &= \sum_{q \leq y} q \sum_{K \in L(X), N_{K,C}=q} 1 \\ &= A(r_2)X \sum_{q \leq y} \frac{q(|C|/|S_n|)}{1+f(q)} \prod_{p < q} \frac{1 - |C|/|S_n| + f(p)}{1+f(p)} + O(y^2 X^{\frac{1+3\delta}{4}}) \\ &= A(r_2)X \sum_q \frac{q(|C|/|S_n|)}{1+f(q)} \prod_{p < q} \frac{1 - |C|/|S_n| + f(p)}{1+f(p)} + O\left(\frac{X}{\log X}\right). \end{aligned}$$

In order to estimate the second sum  $\sum_{K \in L(X), N_{K,C} > y} N_{K,C}$ , we need GRH, which implies that  $N_{K,C} \ll (\log |d_K|)^2$  (cf. [3]). Then

$$\begin{aligned} \sum_{K \in L(X), N_{K,C} > y} N_{K,C} &\ll (\log X)^2 \sum_{N_{K,C} > y} 1 \ll X(\log X)^2 \prod_{p < y} \frac{1 - |C|/|S_n| + f(p)}{1+f(p)} \\ &\ll X(\log X)^2 \left(\frac{|C|}{|S_n|}\right)^{\pi(y)} = O\left(\frac{X}{\log X}\right). \end{aligned}$$

Hence

**Theorem 7.1.** *Let  $L_n^{(r_2)}(X)$  be the set of  $S_n$ -fields  $K$  of signature  $(r_1, r_2)$  with  $|d_K| < X$ . Assume the counting conjectures (2.2) – (2.3) and the GRH for Dedekind zeta functions. Let  $C$  be a conjugacy class of  $S_n$  and  $N_{K,C}$  be the least prime with  $\text{Frob}_p \in C$ . Then,*

$$\frac{1}{|L_n^{(r_2)}(X)|} \sum_{K \in L_n^{(r_2)}(X)} N_{K,C} = \sum_q \frac{q(|C|/|S_n|)}{1+f(q)} \prod_{p < q} \frac{1 - |C|/|S_n| + f(p)}{1+f(p)} + O\left(\frac{1}{\log X}\right).$$

The tables below show average values of  $N_{K,C}$  for  $S_3$ ,  $S_4$ , and  $S_5$ -fields. The computations are done by PARI. The average values of  $N_{K,C}$  for  $S_3$  are given in [20].

$S_3$	Average of $N_{K,C}$	$S_4$	Average of $N_{K,C}$	$S_5$	Average of $N_{K,C}$
$[e]$	19.79522...	$[e]$	108.71075...	$[e]$	716.34521...
$[(12)]$	5.36802...	$[(12)(34)]$	28.96178...	$[(12)(34)]$	29.19651...
$[(123)]$	8.54472...	$[(1234)]$	12.69279...	$[(123)]$	20.75158...
		$[(12)]$	12.69279...	$[(12)(345)]$	20.75158...
		$[(123)]$	9.098479...	$[(12)]$	47.44681...
				$[(1234)]$	12.88664...
				$[(12345)]$	16.72312...

### 8. UNCONDITIONAL RESULTS ON $N_{K,C}$

We can't remove the GRH hypothesis from Theorem 7.1 for a single conjugacy class. However, we may have an unconditional result for the union of conjugacy classes not contained in  $A_n$ . Let  $C'$  be the union of all the conjugacy classes not contained in  $A_n$  and  $N_{K,C'}$  be the smallest prime for which  $Frob_p \in C'$ .

**Theorem 8.1.** *Assume the counting conjectures (2.2)–(2.3), and Conjecture 4.1 with  $\beta_n > \frac{1}{4\sqrt{e}}$ . Then*

$$\frac{1}{|L_n^{(r_2)}(X)|} \sum_{K \in L_n^{(r_2)}(X)} N_{K,C'} = \sum_q \frac{\frac{1}{2}q}{1+f(q)} \prod_{p < q} \frac{\frac{1}{2}+f(p)}{1+f(p)} + O\left(\frac{1}{\log X}\right).$$

Since the proof is similar to previous ones, we just note unconditional and conditional bounds on  $N_{K,C'}$ . Since  $d_K = d_F m^2$  for some integer  $m$ , if  $Frob_p \in C'$ , then  $p$  is inert in  $F$ . (i.e.,  $\left(\frac{d_F}{p}\right) = \left(\frac{d_K}{p}\right) = -1$ .) Conversely, if  $p$  is inert in  $F$  and  $p \nmid d_K$  (i.e.,  $\left(\frac{d_K}{p}\right) = -1$ ), then  $Frob_p$  is in  $C'$ . Hence,  $N_{K,C'}$  is the smallest prime such that  $\left(\frac{d_K}{p}\right) = -1$ . Hence, by Norton [21],

$$N_{K,C'} \ll |d_K|^{\frac{1}{4\sqrt{e}} + \epsilon}.$$

(Norton's result is valid for imprimitive characters.)

Now, we obtain a conditional bound on  $N_{K,C'}$ .

**Proposition 8.2.** *Let  $F$  be a quadratic number field  $\mathbb{Q}(\sqrt{d_F})$ . Assume that  $L(s, \chi_F)$  is zero-free in  $[1 - \alpha] \times [-(\log |d_F|)^2, (\log |d_F|)^2]$ . Then there is a positive constant  $A$  independent of  $\alpha$  for which there exists a prime  $p$  which is inert and  $p \ll (A\alpha \log |d_F|)^{1/\alpha}$ .*

*Proof.* The proof is essentially the same as that of Proposition 4.2 in [6]. □

If  $K$  has a quadratic resolvent  $F$  for which  $L(s, \chi_F)$  has the desired zero-free region, then by Proposition 8.2,  $N_{K,C'} \ll (A\alpha \log |d_K|)^{1/\alpha}$ . Then by using Conjecture 4.1, Theorem 8.1 follows. Since the assumptions in Theorem 8.1 hold for  $S_3$ -fields, we have

**Corollary 8.3.** *Theorem 8.1 holds unconditionally for  $S_3$ -fields. For  $S_3$ -fields and  $C = [(12)]$ , the average value of  $N_{K,C}$  is  $5.36802\dots$ .*

This was proved in Martin and Pollack under GRH ([20], Theorem 4.8).

**Corollary 8.4.** *Under Conjecture 4.1, Theorem 8.1 holds for  $S_4$  and  $S_5$ -fields. For  $S_4$ -fields and  $C' = [(1234)] \cup [(12)]$ , the average value of  $N_{K,C'}$  is  $5.821569\dots$ . For  $S_5$ -fields and  $C' = [(12)(345)] \cup [(12)] \cup [(1234)]$ , the average value of  $N_{K,C'}$  is  $5.9733589\dots$ .*

## 9. APPENDIX: $S_4$ -FIELDS WITH THE SAME CUBIC RESOLVENT

Given a noncyclic cubic field  $M$ , let

$$\Phi_M(s) = 1 + \sum_{K \in \mathcal{F}(M)} \frac{1}{f(K)^s},$$

where  $d_K = d_M f(K)^2$ , and  $\mathcal{F}(M)$  is the set of all  $S_4$ -fields  $K$  with the cubic resolvent field  $M$ . Let  $\mathcal{L}(M, n^2)$  be the set of quartic fields whose cubic resolvents are isomorphic to  $M$  and whose discriminants are  $n^2 d_M$ , and  $\mathcal{L}_{tr}(M, 64)$  the subset of  $\mathcal{L}(M, 64)$ , where 2 is totally ramified. Define  $\mathcal{L}_2(M) = \mathcal{L}(M, 1) \cup \mathcal{L}(M, 4) \cup \mathcal{L}(M, 16) \cup \mathcal{L}_{tr}(M, 64)$ . By Klüners [15],  $|\mathcal{L}(M, n)| \ll (n^2 |d_M|)^{\frac{1}{2} + \epsilon}$ , hence  $|\mathcal{L}_2(M)| \ll |d_M|^{\frac{1}{2} + \epsilon}$ .

By Theorem 1.4 in Cohen and Thorne [10],

$$\Phi_M(s) = \sum_{i=1}^{|\mathcal{L}_2(M)|+1} \Phi_i(s), \quad \Phi_i(s) = \sum_{n=1}^{\infty} \frac{a_i(n)}{n^s},$$

and  $a_i(n) \leq 3^{\omega(n)} \ll 3^{\frac{\log n}{\log \log n}} \ll n^\epsilon$ . Also Theorem 1.4 in [10] implies

$$\Phi_i(1 + c + it) \ll \left( \frac{\zeta(1+c)}{\zeta(2+2c)} \right)^3 \ll \frac{1}{c^3}.$$

By applying Perron's formula to each  $\Phi_i(s)$  for  $i = 1, 2, \dots, |\mathcal{L}_2(M)| + 1$ , we can obtain that

$$|\{K \in \mathcal{F}(M) \mid f(K) \leq x\}| \ll x(\log x)^4 |d_M|^{\frac{1}{2} + \epsilon}.$$

Hence we have proved

**Proposition 9.1.** *Let  $CR_4(X, M)$  be the set of  $S_4$ -fields  $K$  with the given cubic resolvent  $M$  and  $|d_K| \leq X$ . Then*

$$|CR_4(X, M)| \ll X^{\frac{1}{2}} (\log X)^4 |d_M|^\epsilon,$$

*with an absolute implied constant.*

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